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THE RAYLEIGH-TAYLOR INSTABILITY OF TRANSITION LAYER

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Abstract

New types of symmetry for the Rayleigh equation are found. For small Atwood number, an analytic solution is obtained for a smoothly varying density profile. It is shown that a transition layer with a finite width can undergo some kind of stratification.

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Although Rayleigh had investigated the instability (known now as Rayleigh-Taylor (RT) instability) using an exponential profile of density [1], most investigations have been done so using sharp interfaces [2-6]. In [7] it is carried out making an interesting analogy between the equations which describes the RT instability, and the Schrodinger equation. In that study, an "equivalent potential" is constructed and for well-known density profiles it is shown that the Schrodinger equation with the corresponding "equivalent potential" has the same eigenvalues as the Rayleigh equation.

The physical quantities (e.g.; density, velocity) and their derivatives, generally speaking cannot suffer a jump discontinuity. Therefore, it seems more consistent to attempt to solve the problem for a transition layer of finite thickness and then take the limit when the thickness of this layer Δ tends to zero; i.e. consider the case of a density jump in the limit, $\Delta \to 0$. The solution for this transition layer gives us a more complete physical picture of the instability.

In this paper we consider the RT instability of a transition layer of finite thickness, where the unperturbed density changes continuously from a constant value up to another one. For that, an analytic solution is found, in the limit of small Atwood number. This solution allows us to investigate the $\Delta \to 0$ limit. We find that:

1. For an arbitrary density profile the "equivalent potential" can be presented in compact form. This allowed us to predict a new type of symmetry (in addition to the well-known symmetry of the Rayleigh equation [2]).

2. The depth of the "equivalent potential" well depends on the width of the transition layer and the depth not the *energy* of the potential well undergoes a quantization. For the finite width of the layer the spectrum of eigenvalues is infinite. This conclusion is consistent with the fact that with the increase of the quantum number grows the well depth.

3. The eigenfunctions found here show the exfoliation of the transition layer.

I. Using the linearized equation of motion and the continuity equation with the help of

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incompressibility condition leads one to:

$$\rho_0(z)\frac{\partial^2 v(z,t)}{\partial t^2} = \frac{1}{k^2}\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial z}[\rho_0(z)\frac{\partial v(z,t)}{\partial z}] + gv(z,t)\frac{\partial \rho_0(z)}{\partial z}, \tag{1}$$

where $\rho_0(z)$ describes the equilibrium density profile, $v(z,t)$ is the z component of the perturbed fluid velocity, k is the wave number of the perturbed quantities which are chosen proportional to e^{iky} and g is the constant acceleration along negative z. Then one can find the solution of the above equation using separation of variables:

$$v(z,t) = T(t)\Phi(z). \tag{2}$$

Then Eq. (1) reduces into two equations as follows:

$$T''(t) - \gamma^2 T(t) = 0, \tag{3}$$

$$\Phi''(z) + \frac{\rho_0'(z)}{\rho_0(z)}\Phi'(z) + (-k^2 + \frac{k^2 g}{\gamma^2}\frac{\rho_0'(z)}{\rho_0(z)})\Phi(z) = 0, \tag{4}$$

where γ^2 is the constant of separation, primes denote derivatives with respect to corresponding arguments. The case $\gamma^2 > 0$ corresponds to unstable modes.

The substitution

$$\Phi(z) = \frac{\psi(z)}{\sqrt{\rho_0(z)}} \tag{5}$$

reduces Eq. (4) to the form of the Schrodinger equation as in [7]:

$$\psi''(z) + [-k^2 - V(z)]\psi(z) = 0, \tag{6}$$

with the "equivalent potential",

$$V(z) = \frac{1}{\sqrt{\rho_0(z)}}\{\frac{\partial^2}{\partial z^2} - 2k\Gamma\frac{\partial}{\partial z}\}\sqrt{\rho_0(z)}, \tag{7}$$

where

$$\Gamma = \frac{kg}{\gamma^2}. \tag{8}$$

One can further simplify Eqs. (6) , (7) as follows:

$$V(z) = \frac{S''(z)}{S(z)} - k^2\Gamma^2, \tag{9}$$

$$\frac{\psi''(z)}{\psi(z)} - k^2 = \frac{S''(z)}{S(z)} - k^2\Gamma^2, \tag{10}$$

where

$$S(z) = e^{-k\Gamma z}\sqrt{\rho_0}. \tag{11}$$

This form then allows us to prove the following symmetry properties of the Rayleigh equation for arbitrary density profiles:

a) The substitutions

$$\Gamma \rightarrow -\Gamma \quad \text{and} \quad \rho_0(z) \rightarrow \rho_0(z)e^{-4k\Gamma z} \tag{12}$$

cause Eq. (9), and consequently Eq. (6), to remain unchanged. This means that every unstable mode, $\Gamma > 0$, with the density profile $\rho_0(z)$ can be compared with a stable, oscillating mode, $\Gamma < 0$ (see Eqs. (3), (8)), with the density profile $\rho_0(z)e^{-4k\Gamma z}$ and vice versa. For the modified profile, on the other hand, the expression for the velocity (see Eqs. (2), (5)) will contain an additional factor $e^{2k\Gamma z}$.

b) Performing the substitution $\rho_0(z)$ by $\rho_0(-z)e^{4k\Gamma z}$ and then the transformation $z \rightarrow -z$, we find that the potential (9) and Eq. (6) remain unchanged. Consequently the spectrum of the eigenvalues for both profiles are the same, whereas eigenfunction of the latter can be obtained by sending $z \rightarrow -z$ in the former. Furthermore, the function $\Phi(z)$ upon transformation becomes $e^{-2k\Gamma z}\Phi(z)$. Note that these symmetry properties hold only for fixed $\Gamma_n(n = 0, 1, 2, ...)$

II. Let us assume that the continuous equilibrium density tends to different constant limits, i.e. $\rho_0(z) \rightarrow \rho_1$ at $z \rightarrow -\infty$ and $\rho_0(z) \rightarrow \rho_2$ at $z \rightarrow +\infty$, where $\rho_2 > \rho_1$, and $\rho_0'(\pm\infty) = 0$. Then if somewhere in the region $-\infty < z < +\infty$ the "equivalent potential" becomes negative, according to Eq. (6) the RT problem becomes analogous to a particle's motion in a potential well. Using a tanh function representation for density:

$$\rho_0(z) = c(1 + A \cdot \tanh \frac{2z}{\Delta}), \tag{13}$$

where

$$c = \frac{\rho_2 + \rho_1}{2} \quad , \quad A = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}, \tag{14}$$

enable us to examine different limits, from an smooth to a sharp jump limit. We restrict ourselves to the case of small Atwood number , $A \ll 1$, where one can obtain an analytic

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solution. Neglecting the terms of order A^2 from Eq. (6) we obtain:

$$\psi''(x) + \{-\delta^2 + \frac{1}{\cosh^2 x}[\delta \cdot A\Gamma + A \cdot \tanh x]\}\psi(x) = 0, \tag{15}$$

where $\delta = k\Delta/2$, $x = 2z/\Delta$. We made an assumption that $A\Gamma \geq 1$, which will be confirmed by the results. For a broad transition layer, i.e. $\delta \gg A$, Eq. (15) can be reduced to a form which is well-known in quantum mechanics [8] (see p. 73):

$$\psi''(x) + [-\delta^2 + \frac{\delta \cdot A\Gamma}{\cosh^2 x}]\psi(x) = 0 \tag{16}$$

Solution to this equation finite for $z \rightarrow \pm\infty$, can be expressed in terms of hypergeometric function F:

$$\psi = \cosh^{-\delta} x \ F(\delta - s, \delta + s + 1, \delta + 1, (1 - \tanh x)/2), \tag{17}$$

the parameters of which (δ , $s = \frac{1}{2}[-1 + \sqrt{1 + 4\delta A\Gamma}]$) have to satisfy the relation:

$$\delta - s = -n, \tag{18}$$

where $n = 0, 1, 2, ...$. From Eq. (18) we find the spectrum of growth rates for RT instability:

$$\frac{\gamma^2}{kg} = \frac{\delta}{(\delta + n)(\delta + n + 1)}A. \tag{19}$$

Consequently for a transition layer of finite width, $\delta \approx 1$, there is an infinite spectrum of eigenvalues. Number of the extremum and zero points of the eigenfunctions (17) is equal to $n+1$ and n , respectively. For $n > 0$ the spatial dependence of the eigenfunction has oscillatory character. It means that due to the instability, the monotonous distribution of the density in the transition layer can be destroyed, and a stratified and multi-stream structure can be formed. The velocity changes the direction as one goes from one new-formed sublayer to another. The Figure shows the spatial distribution of the velocity (determined from the function $\Phi(z)$; see Eq. (2)) and the total density $\rho(z) = \rho_0(z) + \tilde{\rho}(z)$ in such a stratified layer. The density perturbation $\tilde{\rho}(z)$ is obtained from the continuity equation:

$$\tilde{\rho}(z) = -\Phi(z)\frac{\partial\rho_0(z)}{\partial z}\frac{1}{\gamma}\int_0^{\gamma t} d\tau T(\tau). \tag{20}$$

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As we are concerned in the spatial dependence only, by plotting the figure we treated the definite integral as constant (for simplicity we choose it as equal to one). If the width of the transition layer is small, say

$$\delta = k\Delta/2 \leq A \ll 1, \tag{21}$$

we can consider such a layer as a jump in density. However, in this case the analytic solution of Eq. (15) fails. It is necessary to note that although the spectrum represented by Eq. (19) is obtained for a broad transition layer, $\delta \gg A$, (when we neglect the second term in the square brackets in Eq. (15)) Eq. (19) gives the correct expression for the main unstable mode in the case of density jump, when $\delta \ll A$. In fact as $\delta \to 0$ from Eq. (19) it follows that the most unstable mode corresponds to the quantum number $n = 0$, and for the growth rate one obtains the well-known expression [1, 2]:

$$\gamma = \sqrt{kgA}. \tag{22}$$

Such a general meaning of Eq. (19) can be explained by the fact that, for $\delta \ll 1$, $A \ll 1$ the "equivalent potential" in Eq. (15) can be considered as a perturbation. The quantum mechanical problem of particle motion in a one-dimensional well, whose depth is small, is solved in [8] (see p. 162). There an equation for the determination of the eigenvalues of the Schrodinger equation is obtained. That equation in our notation becomes:

$$-2\delta = \int_{-\infty}^{+\infty} dx V(x). \tag{23}$$

It is obvious that the second term of Eq. (15) doesn't give any contribution in this integral (this term is odd). The first term gives the correct expression for the growth rate consistent with Eq. (22). That is why for the small "equivalent potential", $(\delta \ll 1, A \ll 1)$ the second term in square brackets in Eq. (15) does not play any role in the determination of the growth rate and the expression (19) is valid for a shallow well also.

We can summarize our results as follows:

1. The new symmetry properties found in this paper show new density profiles for which the problem of instability can be reduced to the one and the same equation [3, 7, 9].

2. According to the symmetry theorem known up to now, the two profiles $\rho_0(z)$ and $1/\rho_0(-z)$ have the same set of eigenvalues but not eigenfunctions [2, 3, 9]. The symmetry properties found here maintain the spectrum of eigenvalues and allow one to determine a relationship between the eigenfunctions (see Eq. (5)).

3. The Eq. (10) shows that the inverse problem (obtaining the density profile for known perturbed quantity) is described by the same type of equation as the original problem.

4. From Eq. (17) it follows that for every eigenmode, there exists a unique transition layer of stratified structure.

5. It should be interesting to extend this analysis when separation of variables is not used and thus the spatial and temporal parts not treated independently.

These results are true not only for the chosen density profile from Eq. (13) but also for any density profile whose corresponding "equivalent potential" has the form of a well.

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